Recursive definitions are familiar in mathematics. For instance, the function f defined by

$$\begin{aligned} f(0) &= 1, \\ f(1) &= 1, \\ f(x+2) &= f(x+1) + f(x), \end{aligned}$$

gives the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ... (The study of *difference equations* concerns the problem of going from recursive definitions to algebraic definitions. The Fibonacci sequence is give by the algebraic definition

$$f(x) = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{x+1} - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{x+1}.)$$

The *primitive recursive functions* are an example of a broad and interesting class of functions that cam be obtained by such a formal characterization.

Definition The class of *primitive recursive functions* is the smallest class $\mathcal{C}$ (i.e., intersection of all classes $\mathcal{C}$) of functions such that

i. All *constant functions*, $\lambda x_1 x_2 \cdots x_k[m]$ are in $\mathcal{C}$, $1 \leq k$, $0 \leq m$;

ii. The *successor function*, $\lambda x[x+1]$, is in $\mathcal{C}$;

iii. All *identity functions*, $\lambda x_1 \cdots x_k[x_i]$ are in $\mathcal{C}$, $1 \leq i \leq k$;

iv. If f is a function of k variables in $\mathcal{C}$, and g_1, g_2, ..., g_k are (each) functions of m variables in $\mathcal{C}$, then the function $\lambda x_1 \cdots x_m[f(g_1(x_1, \ldots, x_m), \ldots, g_k(x_1, \ldots, x_m))]$ is in $\mathcal{C}$, $1 \leq k, m$;

v. If h is a function of $k+1$ variables in $\mathcal{C}$, and g is a function of $k-1$ variables in $\mathcal{C}$, then the unique function f of k variables satisfying

$$\begin{aligned} &f(0, x_2, \ldots, x_k) = g(x_2, \ldots, x_k), \\ &f(y+1, x_2, \ldots, x_k) = h(y, f(y, x_2, \ldots, x_k), x_2, \ldots, x_k) \end{aligned}$$

is in $\mathcal{C}$, $1 \leq k$. (For (v), "function of zero variables in $\mathcal{C}$" is taken to mean a fixed integer.)